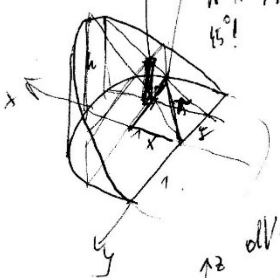
Volumen eines Zylinderhufs & Volumen eines Zylinderkeils

1) Volumen eines Zylinderhufs

Integration via Doppelintegral

$dx \cdot dy$



$\sqrt{r^2-x^2} = \sqrt{1-x^2}$

45°!

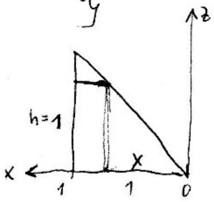
$r = h = 1$!

$x = z$!
$z = x$!

Volumen Zylinderhuf via Doppelintegral:

$$V = \iint_A z\,dA = \int_x\int_y z\,dy\,dx = \int_{x=r-b}^{r}\int_{y=-\sqrt{r^2-x^2}}^{\sqrt{r^2-x^2}} z \cdot dy \cdot dx$$

$$dV = dx\,dy \cdot z(x)$$
$$= dx\,dy \cdot x$$



$$V = \int_{x=0}^{1}\left[\int_{y=-\sqrt{1-x^2}}^{+\sqrt{1-x^2}} x \cdot dy\right] dx = \int_{x=0}^{1} x \left[\int_{y=-\sqrt{1-x^2}}^{\sqrt{1-x^2}} dy\right] dx =$$

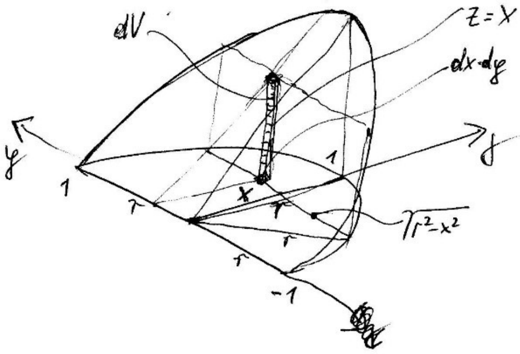
$$= \int_{x=0}^{1} x \cdot y \Big|_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} dx$$

$$= \int_{x=0}^{1} \left[x \cdot \sqrt{1-x^2} - \left(-x \cdot \sqrt{1-x^2}\right)\right] dx$$

$$= \int_{x=0}^{1} 2 \cdot x \cdot \sqrt{1-x^2}\,dx$$

Rechteckfläche!

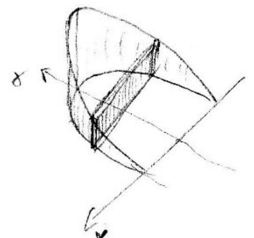
$$= \left[-\frac{2}{3}\left(1-x^2\right)^{3/2}\right]_0^1 = 0 - \left(-\frac{2}{3}\,1\right) = +2/3$$



$z = x$ (45°!) $\Big\}\ dV = x \cdot dx \cdot dy$

$$V_{ZHuf} = \int\limits_{x=0}^{1} \int\limits_{y=-\sqrt{1-x^2}}^{\sqrt{1-x^2}} x \cdot dx \cdot dy =$$

drehen!



Rechtecksfläche $2\sqrt{1-x^2}\cdot x$, dx

$$= \int\limits_{x=0}^{1} \Bigg(\underbrace{y \Big|_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}}_{\sqrt{1-x^2}-\left(-\sqrt{1-x^2}\right) \,=\, 2\cdot\sqrt{1-x^2}} \Bigg) x \cdot dx = \int\limits_{x=0}^{1} 2 \cdot \sqrt{1-x^2} \cdot x\, dx =$$

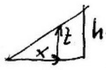
$$= -\frac{2}{3}\left(1-x^2\right)^{3/2}\Big|_0^1 = \left(0-\left(-1^{3/2}\right)\right)\frac{2}{3} = \underline{\underline{+2/3}}$$

allgemein:

$$V_{ZHuf} = \int\limits_{x=0}^{r} \int\limits_{y=-\sqrt{r^2-x^2}}^{+\sqrt{r^2-x^2}} x\, dx\, dy = \int\limits_{x=0}^{r} 2 \cdot \sqrt{r^2-x^2}\, x\, dx = -\frac{2}{3}\left(r^2-x^2\right)^{3/2}\Big|_0^r = 0 - \left(-\frac{2}{3}\left(r^2-0\right)^{3/2}\right) =$$

$$= +\frac{2}{3}\left(r^2\right)^{3/2} = \underline{\underline{\frac{2}{3} r^3}}$$

noch allgemeiner:



$h = p \cdot r$

$z = p \cdot x$

$$V_{ZH} = \int\limits_{x=0}^{r} \int\limits_{-\sqrt{r^2-x^2}}^{+\sqrt{r^2-x^2}} p \cdot x\, dx\, dy = \ldots = \frac{2}{3} r^3 \cdot p = \frac{2}{3} r^2 (r \cdot p) = \underline{\underline{\frac{2}{3} r^2 h}}$$

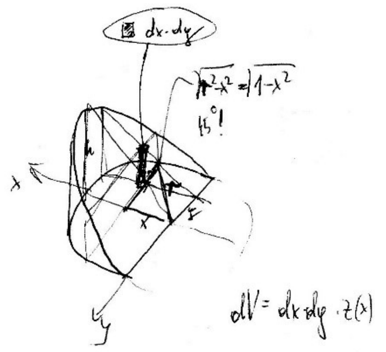
2) Volumen eines Zylinderkeils

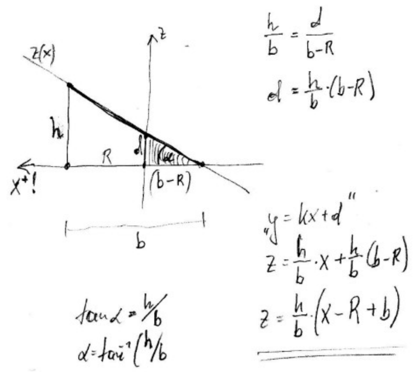
Integration via Doppelintegral

wolframalpha.com verweigert

Input

$$\int_{-b+r}^{r}\int_{-\sqrt{r^2-x^2}}^{\sqrt{r^2-x^2}} \frac{h\,(x-r+b)}{b}\,dy\,dx$$





https://mathdf.com/int/

Integration in zwei Schritten: 1.) dy dann 2.) dx

1.)

h/b(x-r+b) $\displaystyle\int_{-\sqrt{r^2-x^2}}^{\sqrt{r^2-x^2}} \frac{h}{b}\,(x-r+b)\,\mathrm{d}\,y \quad \rightarrow \quad 2\int_0^{\sqrt{r^2-x^2}} \frac{h\,(x-r+b)}{b}\,\mathrm{d}y$ =

= $\dfrac{(2\,h\,x-2\,h\,r+2\,b\,h)\,\sqrt{r^2-x^2}}{b}$

2.)

((2*h*x-2*h*r+2*b*h)*sqrt(r^2-x^2))/b $\displaystyle\int_{r-b}^{r} \frac{(2\,h\,x-2\,h\,r+2\,b\,h)\,\sqrt{r^2-x^2}}{b}\,\mathrm{d}x$ =

$$= \frac{(6\,h\,r^3-6\,b\,h\,r^2)\,\arcsin\left(\frac{r-b}{r}\right)-3\,\pi\,h\,r^3+\sqrt{2\,b\,r-b^2}\,\left(6\,h\,r^2-4\,b\,h\,r+2\,b^2\,h\right)+3\,\pi\,b\,h\,r^2}{6\,b}$$

$r>0, \quad r-b<r$

Vergl. mit identem Resultat in ZKeil_Integration_via_Rechteck_Schnitte.pdf:
Formel umgestellt:
h/(6b)*(6*r^2(r-b)*asin((r-b)/r)+3*pi*r^2(b-r)+sqrt(b*(2r-b))*(6*r^2-4*b*r+2*b^2))

$$\frac{h}{6\cdot b}\cdot\left(6\cdot r^2\cdot(r-b)\cdot\arcsin\left(\frac{r-b}{r}\right)+3\cdot\pi\cdot r^2\cdot(b-r)+\sqrt{b\cdot(2\cdot r-b)}\cdot\left(6\cdot r^2-4\cdot b\cdot r+2\cdot b^2\right)\right)$$

bzw.

$$\boxed{V=\frac{h}{6\,b}\,\left(6\,r^2\,(r-b)\,\arcsin\left(\frac{r-b}{r}\right)+3\,\pi\,r^2\,(b-r)+\sqrt{b\,(2\,r-b)}\,\left(6\,r^2-4\,b\,r+2\,b^2\right)\right)}$$

Nota bene: für b = r ist (b – r) = 0 womit das π „verschwindet“ !