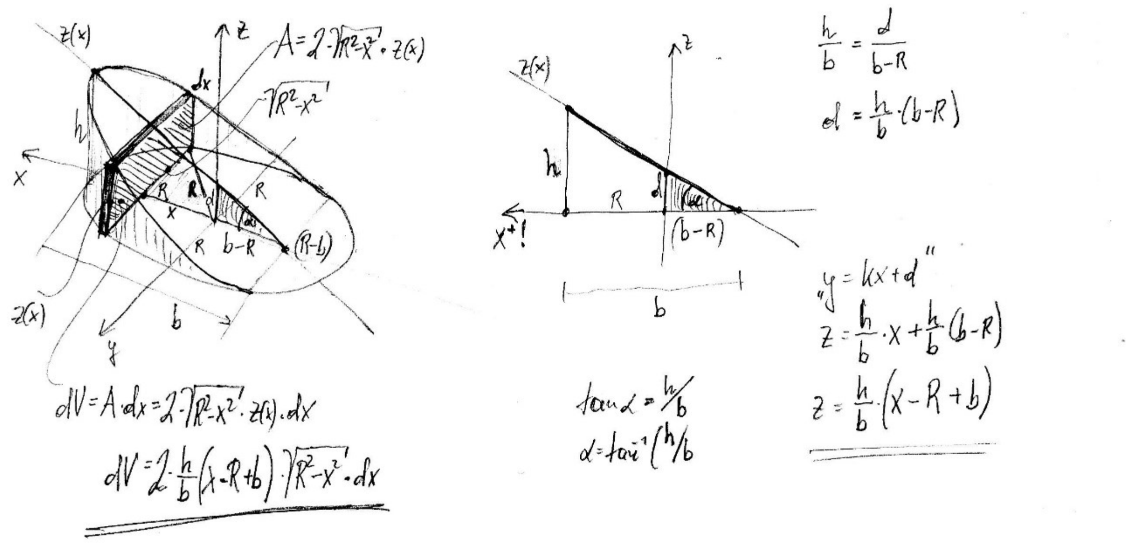
Volumen eines Zylinderkeils (allgemein)
Integration via Rechteck-Schnitte



https://mathdf.com/int/
2*h/b*(x-r+b)sqrt(r^2-x^2)
integrate((2*h*(x-r+b)*sqrt(r^2-x^2))/b,x,r-b,r)

$$\int_{r-b}^{r} \frac{2\,h\,(x-r+b)\,\sqrt{r^2-x^2}}{b}\,\mathrm{d}x$$

$$\frac{\left(6\,h\,r^3-6\,b\,h\,r^2\right)\,\arcsin\left(\frac{r-b}{r}\right)-3\,\pi\,h\,r^3+\sqrt{2\,b\,r-b^2}\,\left(6\,h\,r^2-4\,b\,h\,r+2\,b^2\,h\right)+3\,\pi\,b\,h\,r^2}{6\,b}$$

$r>0,\quad r-b<r$

Formel umgestellt:

h/(6b)*(6*r^2(r-b)*asin((r-b)/r)+3*pi*r^2(b-r)+sqrt(b*(2r-b))*(6*r^2-4*b*r+2*b^2))

$$\frac{h}{6\cdot b}\cdot\left(6\cdot r^2\cdot(r-b)\cdot\arcsin\left(\frac{r-b}{r}\right)+3\cdot\pi\cdot r^2\cdot(b-r)+\sqrt{b\cdot(2\cdot r-b)}\cdot\left(6\cdot r^2-4\cdot b\cdot r+2\cdot b^2\right)\right)$$

bzw.

$$\boxed{V=\frac{h}{6\,b}\,\left(6\,r^2\,(r-b)\,\arcsin\left(\frac{r-b}{r}\right)+3\,\pi\,r^2\,(b-r)+\sqrt{b\,(2\,r-b)}\,\left(6\,r^2-4\,b\,r+2\,b^2\right)\right)}$$

Nota bene: für b = r ist (b – r) = 0 womit das π „verschwindet" !

Vergl. mit Ergebnis der Kreis-Segment-Integration:

$$V_{ZKeil}=\int_{z=0}^{z=h}A_S(z)dz=\frac{h}{3b}\cdot\left(3r^2(b-r)\arccos\left(\frac{r-b}{r}\right)+\sqrt{b(2r-b)}(b^2-2br+3r^2)\right)$$

händische Nachkalkulationen:

$$V = 2\int_{R-b}^{R} \frac{h(x-R+b)}{b}\sqrt{R^2-x^2}\,dx$$

$$= 2\cdot\frac{h}{b}\left[\int_{R-b}^{R} x\cdot\sqrt{R^2-x^2}\,dx + (b-R)\cdot\int_{R-b}^{R}\sqrt{R^2-x^2}\,dx\right]$$

$$= 2\frac{h}{b}\left[\left(-\frac{1}{3}\right)\left(R^2-x^2\right)^{3/2}\Big|_{R-b}^{R} - (R-b)\left(\frac{1}{2}x\sqrt{R^2-x^2}+\frac{R^2}{2}\arcsin\left(\frac{x}{R}\right)\right)\Big|_{R-b}^{R}\right]$$

$$\underbrace{\sqrt{}\cdot\sqrt{}\cdot\sqrt{}}_{(R^2-x^2)\cdot\sqrt{R^2-x^2}}$$

$$= 2\frac{h}{b}\left[\left[\frac{1}{3}\cdot\frac{2}{6}(x^2-R^2)\sqrt{R^2-x^2}+(b-R)\frac{3}{6}\frac{1}{2}x\sqrt{R^2-x^2}\right]_{R-b}^{R} - \left[\frac{1}{2}(R-b)\cdot R^2\arcsin\left(\frac{x}{R}\right)\right]_{R-b}^{R}\right]$$

$$= 2\frac{h}{b}\left[\left[\frac{1}{6}\sqrt{R^2-x^2}\left(2x^2-2R^2+3x(b-R)\right)\right]_{R-b}^{R} - \left[\quad\sim\quad\right]_{R-b}^{R}\right]$$

$$\left(2x^2+3(b-R)x-2R^2\right)$$

also die STAMMFUNKTION

$$= 2\frac{h}{b}\left[\frac{1}{6}\sqrt{R^2-x^2}\cdot\left(2x^2+3(b-R)\cdot x-2R^2\right) - \frac{1}{2}(R-b)\cdot R^2\cdot\arcsin\left(\frac{x}{R}\right)\right]\Bigg|_{R-b}^{R}$$

$$+\frac{1}{2}(b-R)\cdot R^2\arcsin\left(\frac{x}{R}\right)$$

zB: $R=1$

$$\text{zB.}\quad = 2\frac{h}{b}\left[\frac{1}{6}\sqrt{1-x^2}\,\left(2x^2+3(b-1)\cdot x-2\right) + \frac{1}{2}(b-1)\cdot\arcsin(x)\right]_{1-b}^{1}$$

$$= 2\frac{h}{b}\left[\underbrace{\overset{OG}{\frac{1}{6}\sqrt{R^2-R^2}\cdot(\sim)}}_{0} - \overset{UG}{\frac{1}{6}\underbrace{\sqrt{R^2-(R-b)^2}}_{\sqrt{R^2-R^2+2Rb-b^2}}\cdot\left(2(R-b)^2+3\underbrace{(b-R)}_{-(R-b)}(R-b)-2R^2\right)}\right.$$

$$\overset{OG:}{+\frac{1}{2}(b-R)R^2\arcsin\underbrace{\left(\frac{R}{R}\right)}_{\frac{\pi}{2}}} - \overset{UG:}{\frac{1}{2}(b-R)R^2\arcsin\left(\frac{R-b}{R}\right)}$$

$$-\frac{1}{6}\sqrt{2bR-b^2}\left(2(R^2-2bR+b^2)-3(R^2-2bR+b^2)-2R^2\right)$$

$$2R^2-4bR+2b^2-3R^2+6bR-3b^2-2R^2$$

$$\left(-3R^2+2bR-b^2\right)$$

$$+\frac{1}{6}\left(3R^2-2bR+b^2\right)$$

$$\left.+\frac{1}{4}\pi(b-R)R^2 + \frac{1}{2}(R-b)R^2\arcsin\left(\frac{R-b}{R}\right)\right]$$

$$= 2\frac{h}{b}\cdot\frac{1}{12}\left\{\sqrt{2bR-b^2}\cdot\left(6R^2-4bR+2b^2\right) + 3\pi(b-R)R^2 + 6(R-b)R^2\cdot\arcsin\left(\frac{R-b}{R}\right)\right\}$$

Stammfunktion + $R = 1$

$$V = 2\frac{h}{b}\left[\frac{1}{6}\sqrt{1-x^2}\left(2x^2+3(b-1)x-2\right)+\frac{1}{2}(b-1)\arcsin(x)\right]_{1-b}^{1}$$

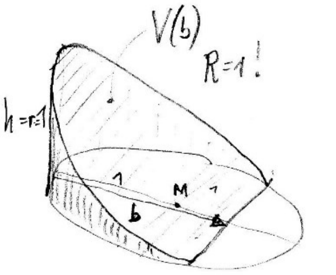
$$= 2\frac{h}{b}\Bigg[\underbrace{\overbrace{\frac{1}{6}\sqrt{1-1}(\dots)}^{OG}}_{\emptyset} - \overbrace{\frac{1}{6}\sqrt{\underbrace{1-(1-b)^2}_{1-1+2b-b^2}}}^{UG}\Big(\underbrace{2(1-b)^2}_{2(1-2b+b^2)}+\underbrace{3\overbrace{(b-1)(1-b)}^{-(1-b)(1-b)}}_{\substack{-3(1-b)^2\\-3(1-2b+b^2)}}-2\Big)+\overbrace{\frac{1}{2}(b-1)\underbrace{\arcsin(1)}_{\pi/2}}^{OG}-\frac{1}{2}(b-1)\arcsin(1-b)$$

$$\left(-\frac{1}{6}\right)\sqrt{2b-b^2}\left(2-4b+2b^2-3+6b-3b^2-2\right) + \frac{1}{4}\pi(b-1)+\frac{1}{2}(1-b)\arcsin(1-b)$$

$$\left(-\frac{1}{6}\right)\sqrt{2b-b^2}\left(2b-b^2-3\right)$$

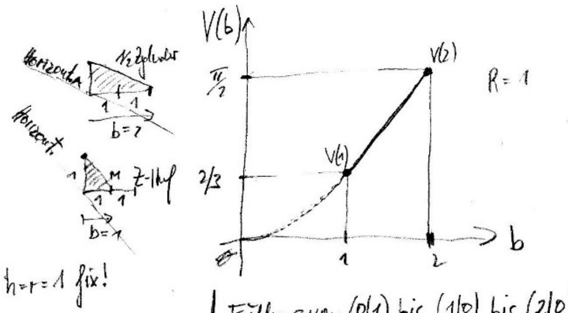
$$= 2\cdot\frac{h}{b}\cdot\left[+\frac{1}{6}\sqrt{2b-b^2}\left(3-2b+b^2\right)+\frac{1}{4}\pi(b-1)+\frac{1}{2}(1-b)\arcsin(1-b)\right]$$

$$= 2\cdot\frac{h}{b}\cdot\frac{1}{12}\left[\sqrt{2b-b^2}\left(6-4b+2b^2\right)+3\pi(b-1)+6(1-b)\cdot\arcsin(1-b)\right]$$



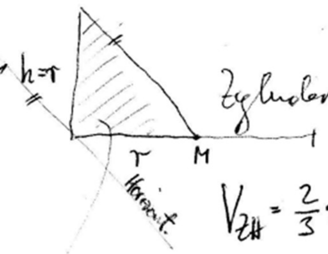
$b=2$: ½ Zylinder $\frac{1}{2}r^2\pi\cdot(h=r) = \frac{1}{2}1^2\pi\cdot 1 = \frac{\pi}{2}$ ✓

$b=1$: Zylinderhuf $\frac{2}{3}r^2\cdot r = \frac{2}{3}$ ✓



! Füllung von $(0|1)$ bis $(\underset{b}{1}|0)$ bis $(\underset{b}{2}|0)$!

fixes Volumen



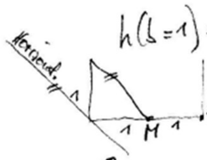
Zylinderhuf

$$V_{ZH} = \frac{2}{3} r^2 \cdot h \overset{(h=r)}{=} \frac{2}{3} r^3$$

$R = 1 \quad V_{ZH} = \frac{2}{3}$

$$\frac{2}{3} = 2 \cdot \frac{h}{b} \cdot \frac{1}{12} \left[\sim + \sim + \sim \right]$$

$$\frac{12}{3} = 4 = \frac{h}{b} \cdot \left[\sim + \sim + \sim \right]$$



$$h(b=1) = \frac{4 \cdot 1}{\sqrt{1}\underbrace{(6-4+2)}_{4} + 3\pi\underbrace{(1-1)}_{0} + 6\underbrace{(1-1)\arcsin(0)}_{\emptyset}} = \frac{4}{4} = 1 \checkmark$$

$$h(b=2) = \frac{8}{0 \cdot (\sim) + 3\pi + \underbrace{6(-1)\arcsin(-1)}_{+3\pi}} = \frac{8}{6\pi} = \frac{4}{3 \cdot \pi}$$

$\arcsin(-1) = -90° = -\frac{\pi}{2}$

$$h(b) = \frac{4b}{\left[\sqrt{2b-b^2}\,(6-4b+2b^2) + 3\pi(b-1) + 6(1-b) \cdot \arcsin(1-b)\right]}$$