HARRY WHITE

THE LOGARITHMIC SPIRAL CURVE

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KEUFFEL & ESSER CO.

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THE LOGARITHMIC SPIRAL CURVE.

BY WM. COX.

INTRODUCTION.

LOGARITHMS are a series of numbers in Arithmetical Progression corresponding to another series of numbers in Geometrical Progression, thus:—

0, 1, 2, 3, 4, 5, etc.,
0, 5, 10, 15, 20, 25, etc.,

are series in *A. P.*, the *difference* between any two consecutive terms of each series being 1 and 5 respectively; and

1, 2, 4, 8, 16, 32, 64, etc.,
1, 3, 9, 27, 81, 243, 729, etc.,
1, 1.5, 2.25, 3.375, etc.,

are series in *G. P.*, the *ratio* of any one term to the succeeding one being as 1:2, 1:3, 1:1.5 respectively.

If we lay off a series of numbers in *A. P.* and underneath it a series in *G. P.*, thus:

0,	5,	10,	15,	20,	25,
1,	1.5,	2.25,	3.375,	5.0625,	7.59375

then the numbers 0, 5, 10, etc., are the logarithms of the corresponding numbers 1, 1.5, 2.25, etc.

The general law of a Geometrical Progression being that the consecutive terms of the series always proceed in the same given ratio, it naturally follows that any four consecutive terms form a proportion, as 1 : 2 :: 4 : 8 or 1 : 3 :: 9 : 27, whence we obtain 1 : 4 :: 2 : 8 or 1 : 9 :: 3 : 27.

We may continue these proportions to any extent, as

1 : 2 :: 4 : 8 :: 16 : 32 :: 64 : 128, etc.,

and if the same ratio between the pairs of terms be observed throughout, we may also obtain such proportions as

2 : 4 :: 8 : 16 :: 32 : 64, etc.
2 : 4 :: 3 : 6 :: 5 : 10 :: 6 : 12, etc.
2 : 8 :: 4 : 16 :: 32 : 128, etc.
2 : 8 :: 3 : 12 :: 5 : 20 :: 6 : 24, etc.

These proportions may be varied infinitely, provided always that the relative and corresponding terms of the first ratio be equally observed in the terms of the second ratio; thus, if *consecutive* terms of the series be taken in the first ratio, consecutive terms must be taken in the second; if *alternate* in the one case, they must be alternate in the other, and so on.

We may also take *three* terms of the series (providing that their positions in the series be similar) and make of them a proportion, thus:

$$4:8::8:16 \text{ or } 2:8::8:32,$$

in which case the middle term is the geometric mean or mean proportional between the two outer terms, its square being equal to the product of the other two terms, that is:

$$4 \times 16 = 8^2 \text{ and } 2 \times 32 = 8^2.$$

As we can graphically represent numbers by lines of given lengths (corresponding with the scale adopted) so we can represent a series of numbers in *A. P.* by a series of lines, and similarly a series of numbers in *G. P.* by another series of lines. The former will vary amongst themselves by a given uniform rate of *difference*, while the latter will vary by a given constant *ratio*.

This is the principle on which the Slide Rule (which is a logarithmic rule) is constructed, and the same principle underlies likewise the construction of the logarithmic curve.

CONSTRUCTION.

On the straight line XZ, Fig. 1, with centre *P*, and any radius *PA*,

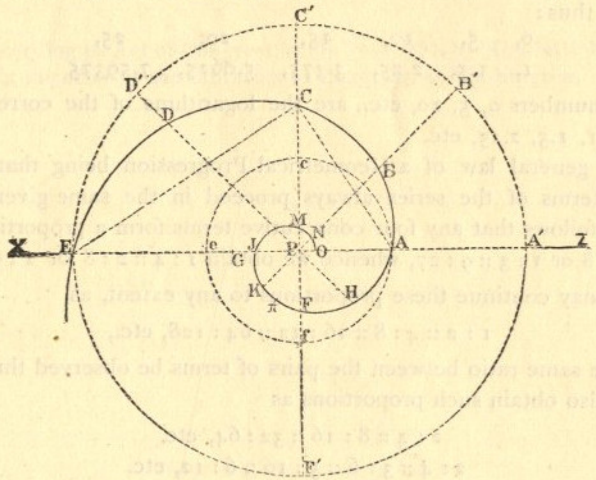

Fig. 1.

draw the circle *Acef*; and with the same centre *P*, and any other greater radius *PE*, draw the circle $A^1C^1EF^1$. We have thus the two radii *PA* and *PE*, between which a certain ratio is established. On these two radii we shall proceed to construct our logarithmic curve, commencing at *A* and ending at *E*; that is to say, a curve must be traced with centre *P*, whose first radius *PA*, shall be that of the small circle *Acef*, and whose other radii shall continually increase, and that uniformly in the *same ratio*, and in such measure, that the last one, PE, shall be also the radius of the larger circle $A^1 C^1 E F^1$. The various radii* of this curve will then represent the series of numbers in Geometrical Progression, whilst the arc $A^1 C^1 E$, *equally* divided in as many points as we have radii, represents the series of numbers in Arithmetical Progression, consequently the different radii must be capable of the same functions which we have shown appertain to a series of numbers in $G. P.$ If, for instance, we divide the semi-circle $A^1 C^1 E$ into 10 equal parts, and draw radii to the centre P, these will indicate on the arc the different terms of a series of numbers in $A. P.$, and the logarithmic curve must so cut them, that the length of the different radii from the centre P to the curve, will indicate the successive terms of a series of numbers in $G. P.$ The lengths of these radii must be so proportioned, that the *same ratio* shall exist throughout between *any two consecutive* radii.

Let us begin by dividing the arc $A^1 C^1 E$ into two equal parts, by drawing a radius from C^1 to the centre P perpendicular to A^1E, so that the angles A^1PC^1 and C^1PE shall be equal to each other; then the logarithmic curve must cut this radius in such a point C that there may be the same ratio between PA and PC, that there is between PC and PE, that is to say, PC will be a mean proportional between PA and PE, thus:

$$PA : PC :: PC : PE$$

$$\text{whence } PA \times PE = PC \times PC = PC^2$$

$$\text{and } PC = \sqrt{PA \times PE}$$

Similarly, if we bisect the angle A^1PC^1, and draw he radius PB^1, the logarithmic curve must also cut this radius in such a point B that

$$PA : PB :: PB : PC$$

$$\text{and } PA \times PC = PB^2,$$

so also by bisecting the angle C^1PE in D^1. we obtain

$$PC : PD :: PD : PE$$

$$\text{and } PC \times PE = PD^2.$$

We have thus five lines of *increasing* length, PA, PB, PC, PD and PE, representing 5 terms of a series of numbers in Geometrical Progression, whose logarithms are represented by A^1, A^1B^1, A^1C^1, A^1D^1 and A^1E, which are the terms of a series of numbers in Arithmetical Progression;

*Called *Radii Vectores* when speaking of curves other than a true circle, as they are not the true radii of curvature of the spiral, which is shown by SK in Fig. 2. For the sake of brevity and to avoid frequent repetition, we have however simply used the term radius.

and in accordance with the laws which govern numbers in *G. P.*, we have

$$PA : PB :: PC : PD \text{ and } PA \times PD = PB \times PC$$
$$PB : PC :: PD : PE \quad \text{"} \quad PB \times PE = PC \times PD$$
$$PA : PC :: PC : PE \quad \text{"} \quad PA \times PE = PC^2$$
$$PA : PB :: PB : PC \quad \text{"} \quad PA \times PC = PB^2$$
$$PA : PD :: PB : PE \quad \text{"} \quad PA \times PE = PD \times PB$$

etc. etc.

Whence it follows that *if an angle, however large or small, formed by any two radii be bisected, then the central radius is a mean proportional between the outer radii, or the square of the central radius is equal to the product of the two extremes.*

We may now in the same manner continue this curve backwards from *A* through *H*, *F* and *K* to *J*, and further through *M* and *N* to *O*, always taking care that the same ratio exists between any two radii enclosing equal angles. The curve might also be continued onwards in the same way from *E*, thus making a spiral of indefinite length. We thus obtain an endless number of similar ratios, out of which we can form an infinite number of proportions, as for example

$$PO : PN :: PA : PB \text{ or } PO : PA :: PN : PB$$
$$PO : PJ :: PH : PD \text{ or } PO : PK :: PB : PE$$

etc., etc.,

in which the two radii or terms of each ratio of a proportion enclose equal angles.

A practical method of obtaining a mean proportional between two radii such as *PA* and *PE*, is to bisect *AE* in *G*, then with centre *G* and radius *GA*, describe a small arc cutting the radius PC^1 of the circle $A^1 C^1 E F^1$ in *C*. This point of intersection is also the point where the logarithmic curve cuts the radius PC^1. This will be evident if we join *AC* and *EC*, as we thus obtain three similar triangles *ACE*, *CPE* and *APC*, whence $PA : PC :: PC : PE.$

By the same law of similar triangles, a line drawn from *A* parallel to *EC* will cut the radius PF^1 in *F*, and another line drawn from *F* parallel to *AC* will cut the radius *PE* in *J*, whilst similar lines from *J* and *M* will cut the radii *PC* and *PA* in *M* and *O*. These points *F*, *J*, *M*, and *O* are likewise all points in the logarithmic curve. The radii *PF*, *PJ*, *PM* and *PO* may be also very easily obtained by means of proportional dividers by setting them to the ratio existing between the two radii *PA* and *PC*. If the radius *PB* be now calculated as a mean proportional between *PA* and *PC*, the other radii *PD*, *PH*, etc., can be laid off in the same way with the proportional dividers; thus by repeatedly bisecting an angle, any number of radii of the logarithmic curve may be easily found.

USES OF THE CURVE.

PROPORTION. Having thus obtained an indefinite number of ratios, it will be readily seen that if we assign to any radius, say PO, a value of unity, and make to this basis a scale of equal parts, we can find the value of any other radii, and having three such values, can obtain that of a fourth proportional.

Let us suppose that we have traced a curve, of which the radius PO is, by a given scale, equal to 10, and the radius PJ is, by the same scale, equal to 31.416, it will naturally follow that *any two* radii forming, as do PO and PJ, an angle of 180 degrees, will be to each other as the diameter of a circle is to its circumference. If the curve be constructed to any other ratio, the angle can be easily found which will give this same ratio of 1 to 3.1416, and by noting this angle*, it will serve at any time to find graphically the circumference of a circle when the diameter is known. (The ratio 113:355 is more correct than that of 1:3.1416, and will be more easily set off with a scale of equal parts). The same may be done for any other ratios, which are of frequent occurrence in the engineer's practice.

As multiplication and division are but other forms of proportion, it naturally follows that this curve may be used for the graphical performance of such calculations.

MULTIPLICATION. We reduce this to the form

$$1 : a :: b : x$$

where a and b are the two factors and x is their product. We then take any radius, say PF, to be by any convenient scale equal to unity, (or any decimal multiple of unity as 10 or 100) and any other radius, say PH, to be equal to the factor a. If radius PC be by the same scale equal to the other factor b, then the fourth radius or x should enclose with PC an angle equal to FPH. Such in Fig. 1, is angle DPC, therefore PD multiplied by unity or the decimal multiple of unity assigned to PF, is equal to the product $PC \times PH$, because

$$PF : PH :: PC : PD$$

$$\text{therefore } PF \times PD = PH \times PC.$$

EXAMPLE. Multiply 12.4 by 23.7.

Let by a given scale $PF = 10$, $PH = 12.4$, and $PC = 23.7$, then, as the angles FPH and CPD are equal,

$$PF : PH :: PC : PD,$$

and by the same scale we find that

$$PD = 29.37, \text{ then } PH \times PC = PD \times PF = PD \times 10 = 293.7 =$$
$$= 12.4 \times 23.7,$$

*NOTE.—For our logarithmic curve, PO being equal to 10 mm., the radius which with radius PO would enclose an angle of 239°4′, will be equal to 31.416 mm.—This radius is marked Fig. 1, π.

or we may proceed as in Fig. 2. Let it be required to multiply 78 by 145 3.

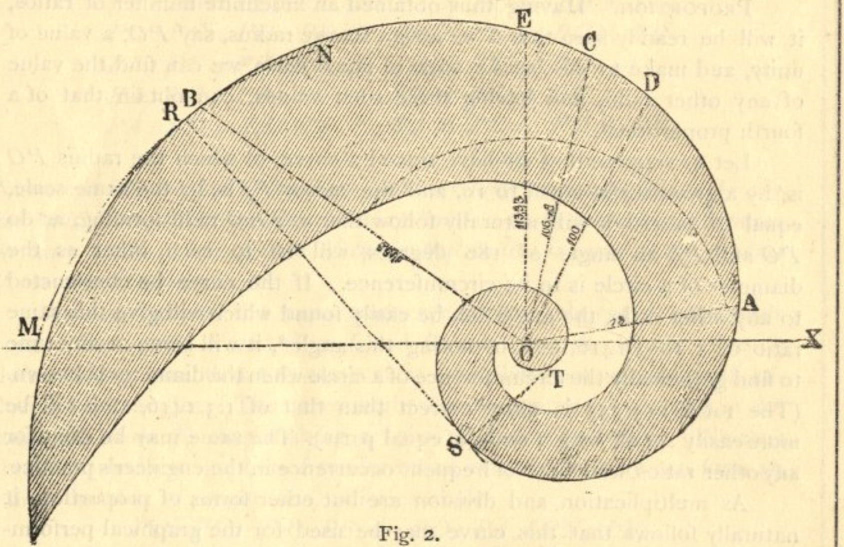

Fig. 2.

Take OA to be, according to any suitable scale, equal to the factor 78, and OB to be by the same scale equal to the factor 145·3. Bisect the angle AOB so that angle AOC = angle COB, then $OC^2 = OA \times OB$.

Take by the same given scale $OD = 100$, and make angle EOC equal to angle COD, then we have

$$OC^2 = OE \times OD \text{ and consequently}$$
$$OA \times OB = OE \times OD$$
$$= OE \times 100 = 113{\cdot}33 \times 100$$
$$= 11333 = 78 \times 145{\cdot}3$$

Division. We reduce this operation to the form

$$a : b :: 1 : x,$$

where a is the divisor, b the dividend and x the quotient. Let, Fig. 1, PE be the dividend, PD the divisor and PF unity, 10 or 100, all by the same scale; then the angle FPH being equal to the angle DPE

$$PD : PE :: PF : PH$$

$$\text{whence } \frac{PE}{PD} = \frac{PH}{PF}$$

that is, PH divided by unity or any other decimal multiple of unity assigned to PF, will be the quotient of PE divided by PD.

Or we may proceed as shown in Fig. 2, where 11333 ÷ 78 is sought. Let $OE = 113.33$ and $OD = 100$; bisect the angle DOE so that angle

DOC will be equal to angle COE, then make $OA = 78$, and with angle BOC equal to angle AOC, lay off OB, then

$$OD : OB :: OA : OE$$

$$\text{whence } \frac{OE}{OA} = \frac{OB}{OD}$$

$$= \frac{145.3}{100} = 1.453$$

but as we have taken $OD = 100$ instead of unity, therefore

$$\frac{OE}{OA} = 1.453 \times 100 = 145.3.$$

The application of the remarkable properties of the logarithmic spiral curve requires that it be constructed with extreme accuracy. We offer therefore to the public such an instrument as is represented in Fig. 2, which will be found to be both durable and correct. We append a table of the different values of the radii of our curve from PO round through PJ and PA to PE (Fig. 1), for a series of angles, advancing 5 degrees at a time, the radius PO being equal to 10 mm., $PJ = 23.68$ mm., $PA = 56.08$ mm. and $PE = 132.80$ mm. The ratio between PO and PJ or PJ and PA or PA and PE is 10 to 23.68, and the ratio between any two radii enclosing an angle of 5 degrees is as 10 to 10.24235.

Degrees in Angle	Radius Vector	Degrees in Angle	Radius Vector	Degrees in Angle	Radius Vector	Degrees in Angle	Radius Vector	Degrees in Angle	Radius Vector
First Turn.									
0	10.00	115	17.35	230	30.09	345	52.19	85	84.26
5	10.24	120	17.77	235	30.82	350	53.46	90	86.30
10	10.49	125	18.20	240	31.56	355	54.75	95	88.39
15	10.74	130	18.64	245	32.33	360	56.08	100	90.53
20	11.01	135	19.09	250	33.11			105	92.73
25	11.27	140	19.55	255	33.92	*Second Turn.*		110	94.97
30	11.55	145	20.03	260	34.74	0	56.08	115	97.27
35	11.82	150	20.51	265	35.58	5	57.44	120	99.63
40	12.11	155	21.01	270	36.44	10	58.83	125	102.05
45	12.41	160	21.51	275	37.32	15	60.26	130	104.52
50	12.71	165	22.04	280	38.23	20	61.86	135	107.05
55	13.01	170	22.57	285	39.16	25	63.21	140	109.95
60	13.33	175	23.12	290	40.11	30	64.74	145	112.31
65	13.65	180	23.68	295	41.08	35	66.31	150	115.03
70	13.98	185	24.25	300	42.07	40	67.92	155	117.82
75	14.32	190	24.84	305	43.09	45	69.57	160	120.67
80	14.67	195	25.44	310	44.13	50	71.25	165	123.59
85	15.02	200	26.06	315	45.21	55	72.98	170	126.59
90	15.39	205	26.69	320	46.30	60	74.75	175	129.66
95	15.76	210	27.34	325	47.42	65	76.56	180	132.80
100	16.14	215	28.00	330	48.57	70	78.41		
105	16.53	220	28.68	335	49.75	75	80.32		
110	16.94	225	29.38	340	50.96	80	82.26		

Some of the other uses of this curve are :

1. The curvature gradually changing, it is peculiarly adapted for fitting to any part of a given curve.

2. It assists in the rapid determination of the centre of curvature of a given curve. If, for instance, the instrument coincides with the curve *MN* at *R*, Fig. 2, then the line *RS*, which is a tangent to the spiral at *S*, gives the direction of the true radius of curvature of the point *R* of the curve *MN*, and the tangent *ST*, at right angles to *RS*, gives the exact position of the centre *S* of curvature.

3. The determination of the centres of curvature gives a convenient means for drawing equi-distant curves.

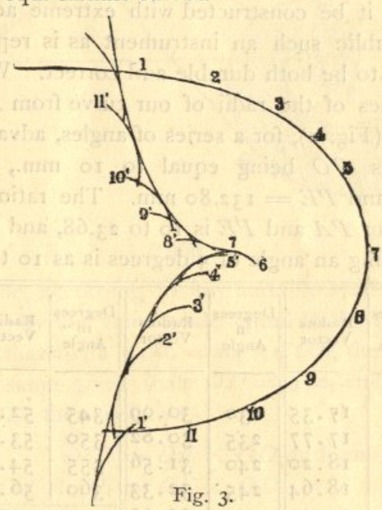

Fig. 3.

4. For drawing the evolute of a curve, (the curve *FK*, Fig. 1, is the evolute of the curve *DE*) the instrument is fitted to a series of points (1, 2, 3, 4) of the curve, each time drawing by the instrument a short piece of the curve containing the centre of curvature (1^1, 2^1, 3^1, 4^1 . . .), which pieces then surround the evolute of the given curve.

If, for any purpose, it should be necessary to continue the curve of the instrument, it can be done by holding a string or wire against the periphery of the instrument, with a pencil at its end, the point of which touches the point *R* of the curve (Fig. 2) while the string takes the line *RS* and continues round the curve to *A*, *D*, *C*, *E*, etc. If the pencil is now moved forward towards *M*, it will mark, as the string unwinds from *S* towards *A*, the continuation of the curve.

The inner curve of the instrument is for drawing curves, of which the ratio between *PA* and *PE* (Fig. 1) is much greater than that of the outer curve, as the size of the spiral for this ratio would be too large.